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On a generalization of Kaden's problem

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Kaden's problem of the roll-up of an initially planar semi-infinite vortex sheet with a parabolic distribution of circulation is extended to include vortex sheets exhibiting a general power law circulation distribution, resulting in the presence of a power law, and in one case a logarithmic-like, velocity-field singularity. Both semi-infinite and infinite initially plane sheets with this property are considered and the form of their roll-up in the similarity plane, into single and double-branched spirals respectively, is obtained numerically. Estimates of the Betz constant obtained from the solutions are found to be significantly different from values predicted by the Betz approximation.

1. Introduction

The evolution with time of an initially plane semi-infinite vortex sheet with a parabolic distribution of circulation was first treated by Kaden (1931). He postulated that the edge of the initially flat sheet rolled up into a continuous spiral and deduced from dimensional arguments that this spiral must always retain the same shape but must grow with time t as $t^{\frac{2}{3}}$. Kaden further showed that towards the centre of the ever tightening spiral, the circulation distribution became asymptotic to the initial parabolic flat sheet form, but with the spacial ordinate from the sheet edge x replaced by βr, where r is the radius from the spiral centre and β is an unknown dimensionless constant whose value can be determined only by detailed dynamical considerations. The significance of Kaden's analysis is clear when we note that to a good approximation, Kaden's problem describes the inviscid roll-up, near the wing tips, of the vortex sheet shed by an elliptically loaded wing at small incidence in steady level flight. Its usefulness cannot be fully realized, however, until the value of β is known, since β, physically, is a measure of the degree to which successive turns of the innermost spiral are compressed during the roll-up process (Saffman & Baker 1979). It is important to know β since Kaden's asymptotic solution may then be interpreted as the inviscid outer limit for the merged core of the vortex forming during the roll-up process. This is applicable both to laminar (Moore & Saffman 1973) and to turbulent (Phillips 1981) trailing vortices.

The first estimate for β was based on an approximation of Betz (1932), that during roll-up the angular impulse of vorticity, relative to the sheet's centre of vorticity, remains constant, leading to $\beta = \frac{1}{2}$. A more precise estimate has been given by Pullin

(1978), who obtained a numerical solution of the nonlinear integro-differential equation which follows from the application of Kaden's dimensional arguments (in the form of a similarity solution) to the full initial value problem for the sheet motion. This solution confirmed Kaden's postulate that the initially plane vortex sheet rolled up smoothly into a tightly wound spiral that closely approximated the asymptotic form he predicted, even over the outermost turn. It also provided a convincing estimate for the Betz constant, $\beta = 2{\cdot}02$.

But the spanwise circulation distribution on a lifting wing is affected by the wing shape, profile, chord and angle of attack (Prandtl & Tietjens 1957) and is therefore not always elliptic. Moreover, for a wing composed of spanwise elements whose profiles are geometrically similar and at equal angles of attack, it follows that the circulation is proportional to the chord. Thus, we might expect the circulation distribution on a delta wing for example, to be approximately linear. Experimental evidence of a relationship between wing shape and circulation distribution is produced in Küchemann (1953). The pertinent question is can the essence of Kaden's argument for roll up be extended to vortex sheets with other spanwise distributions of circulation? We suggest that it can, provided that the velocity field due to the sheet is singular at some point along the sheet length. Moore & Saffman (1973) first realized this, and went on to find the asymptotic form for the innermost spiral of rolling-up vortex sheets with more general initial spanwise distributions of circulation, viz. $\Gamma \propto |x|^p$, where $0 < p \leqslant 1$. Here, again the circulation distribution in the asymptotic rolled-up vortex core follows by putting $x \equiv \beta_p r$, but as before the precise value of β_p remained unknown, although the Betz approximation indicated that $\beta_p = 1+p$.

The purpose of the present work is to generalize the classical Kaden's problem to the roll-up of both semi-infinite and of infinite sheets with initial circulation distributions of the general form $|x|^p$, and as part of this process we shall estimate β_p. The roll-up of the infinite sheet into a double-branched spiral is of interest for two reasons. First, this case is directly relevant to some experimental studies of trailing vortices in which a 'differential' aerofoil spans the wind tunnel, one half being mounted at an angle of incidence equal and opposite to that of the other (Hoffman & Joubert 1963; Poppleton 1971; Graham, Newman & Phillips 1974). This results in a two-branch stable trailing vortex. Second, it will be shown that a certain limit of the present class of infinite sheets corresponds to the case discussed by Jiminez (1977, 1980) as a possible similarity model for the nonlinear instability observed in plane shear layers. The procedure adopted in the present work closely parallels that of Pullin (1978) henceforth called I, and except where otherwise stated, the notation, formulation and method of solution are as in I.

2. Formulation and solution

2.1. *A generalization of Kaden's problem*

We consider the motion for $t \geqslant 0$ of both semi-infinite plane sheets, given at $t = 0$ by $z = x + 0i$, $-\infty < x \leqslant 0$, and infinite plane sheets in $-\infty < x < \infty$. The initial circulation distributions are given by

$$\left.\begin{aligned} \Gamma &= 2a|x|^p, \quad -\infty < x \leqslant 0, \\ \Gamma &= -2a\delta x^p, \quad 0 \leqslant x < \infty, \end{aligned}\right\} \tag{1}$$

where a is a dimensional constant and p is a dimensionless constant which lies in the range $0 < p \leqslant 1$. The parameter δ takes the value $\delta = 0$ for the semi-infinite sheet and $\delta = 1$ for the infinite sheet. In both cases the strength density $\gamma = -d\Gamma/dx$ exhibits an $|x|^{p-1}$ like singularity at $x = 0$ so that for $\delta = 0$ and 1, the sheet may be expected to roll-up into single-branched and double-branched spirals respectively. For $\delta = 0$ the case $p = \frac{1}{2}$ was treated in I. The limiting case $p = 1$ indicates a linear Γ distribution. Here the γ singularity vanishes but it may be easily shown that the velocity field remains singular as $\log|x|, x \to 0$. The other limiting case $p = 0$ represents a constant Γ distribution which has no strict physical meaning (Moore & Saffman), but a distribution in which $p \to 0$ can be achieved on a rectangular wing of high aspect ratio (Betz 1919). The trailing vortex studied by Hoffman and Joubert corresponds most nearly to $\delta = 1, p = \frac{1}{2}$ while that of Graham *et al.* corresponds nearly to $\delta = 1, p = \frac{1}{4}$. For $\delta = 1$ and $p = 1$ both the γ and the velocity singularities vanish so that the plane sheet is of constant strength everywhere. This is the case studied by Jiminez.

The single branch sheet shape for $\delta = 0$ and the left branch for $\delta = 1$ shall be described by $z_0(\Gamma, t), \Gamma \geqslant 0$. For $\delta = 1$, the right branch is given by $-z_0(\Gamma, t)$, obtained by rotating the left branch through π radians. Thus, two-branch antisymmetry for $\delta = 1$ allows us to deal only with the left branch, affording significant simplification. The equation of motion for $z_0(\Gamma, t)$ is then

$$\left(\frac{\partial \bar{z}_0}{\partial t}\right)_\Gamma = \frac{1}{2\pi i}\int_0^\infty \frac{d\Gamma'}{z_0(\Gamma,t) - z_0(\Gamma',t)} + \frac{\delta}{2\pi i}\int_0^\infty \frac{d\Gamma'}{z_0(\Gamma,t) + z_0(\Gamma',t)}, \quad 0 \leqslant \Gamma < \infty, \tag{2}$$

($\bar{z}_0$ denotes the complex conjugate of z_0), with the initial condition obtained from (1) as

$$z_0(\Gamma, 0) = -(\tfrac{1}{2}\Gamma/a)^{1/p}. \tag{3}$$

The similarity solution takes the form

$$z_0(\Gamma, t) = (at)^{1/(2-p)}\,\omega(\lambda), \tag{4}$$

where $\omega(\lambda) = \xi(\lambda) + i\eta(\lambda)$ is the non-dimensional sheet-shape function and the independent variable λ is given by

$$\lambda = \frac{\Gamma}{a^{2/(2-p)}t^{p/(2-p)}}. \tag{5}$$

The integro-differential equation for $\omega(\lambda)$ then follows from (2) as

$$\frac{1}{2-p}\left(\bar{\omega} - p\lambda\frac{d\bar{\omega}}{d\lambda}\right) = \frac{1}{2\pi i}\int_0^\infty \frac{d\lambda'}{\omega - \omega(\lambda')} + \frac{\delta}{2\pi i}\int_0^\infty \frac{d\lambda'}{\omega + \omega(\lambda')}, \tag{6}$$

with

$$\omega(\lambda) = -\left(\frac{\lambda}{2}\right)^{1/p} + \text{small correction} \tag{7}$$

as $\lambda \to \infty$.

2.2. *Asymptotic behaviour of solutions*

(i) $\lambda \to \infty$. The second term in the expansion given by (7) may be obtained by substituting (7) into the left-hand side of (6) and the first term of (7) into the integrals on the right-hand side. After evaluating the integrals by residues, we find that

$$\omega(\lambda) = -\left(\frac{\lambda}{2}\right)^{1/p} - ip\left[\frac{\delta + \cos(p\pi)}{\sin(p\pi)}\right]\left(\frac{\lambda}{2}\right)^{1-1/p} + \dots \tag{8}$$

as $\lambda \to \infty$. Returning to physical co-ordinates it may be readily verified that the second term in (8) represents the leading-order contribution to the down-wash induced by distant parts of the sheet in the vicinity of the rolled-up spiral. For $\delta = 0$, this contribution is in the $+$ and $-$ η directions for $p > \frac{1}{2}$ and $p < \frac{1}{2}$ respectively, vanishing identically for $p = \frac{1}{2}$, while for $\delta = 1$, this contribution is always in the $-\eta$ direction.

(ii) *Semi-infinite sheet, $p \to 1$.* It is clear that in the limit $p \to 1$, the expansion given by (8) remains regular for $\delta = 1$ but fails for $\delta = 0$. The reason is that the self-inductive velocity of the semi-infinite sheet, given effectively by the integral on the right side of (6), diverges as $ip\cot(p\pi)$ as $p \to 1$. The sheet behaviour in this limit may be nevertheless obtained by introducing the transformation for $\delta = 0$ suggested by (8),

$$\rho = \omega + ip\cot(p\pi), \tag{9}$$

where $\rho = \xi' + i\eta'$. Substituting (9) into (6), adding the identity

$$\frac{1}{2\pi i}\int_0^\infty \frac{d\lambda'}{(\lambda/2)^{1/p} - (\lambda'/2)^{1/p}} + ip\cot(p\pi)\left(\frac{\lambda}{2}\right)^{1-1/p} \equiv 0$$

to the right side of the result, and taking the $p \to 1$ limit leads to

$$\bar{\rho} - \lambda\frac{d\bar{\rho}}{d\lambda} = \frac{1}{2\pi i}\int_0^\infty \left[\frac{1}{\rho - \rho'} + \frac{1}{\lambda/2 - \lambda'/2}\right] d\lambda' + \frac{i}{\pi}\left[\log(\lambda/2) - 1\right]. \tag{10}$$

A solution to (10) may be interpreted as the roll-up of a constant strength ($p = 1$) semi-infinite sheet as seen in a framework moving with the infinite self-inductive velocity due to parts of the sheet remote from the spiral (see appendix for further discussion). The leading terms in the large λ expansion for $\rho(\lambda)$, ($p = 1$), may be obtained either by following the procedure leading to (8) or by direct substitution of (9) into (8), to yield

$$\rho(\lambda) = -(\lambda/2) - \frac{i}{\pi}\log(\lambda/2) + \dots, \tag{11}$$

the order of the limits being understood as $p \to 1$, $\lambda \to \infty$. Equation (11) shows that the asymptotic sheet shape in the ρ plane in this case is given by $\eta' = -\log|\xi'|/\pi$, $\xi' \to -\infty$.

(iii) $\lambda \to 0$. The small λ solution applies in the innermost rolled-up portion of the spiral. Here the leading term in the expansion of the solution to (6) takes the form (see Moore & Saffman 1973; Moore 1975),

$$\omega(\lambda) - \omega(0) = \alpha_p \lambda^{1/p} \exp\left[i\left\{\frac{\lambda^{1-2/p}}{\pi(2-\delta)\alpha_p^2} + \epsilon\right\}\right], \tag{12}$$

where α_p and ϵ are constants to be determined. Equation (12) may be shown to be valid in the limit $p \to 1$ for $\delta = 0$. The Betz constant may be related to α_p by using the general definition given by Moore & Saffman (1973) in conjunction with (12) to yield

$$\beta_p = (2^{1/p}\alpha_p)^{-1}. \tag{13}$$

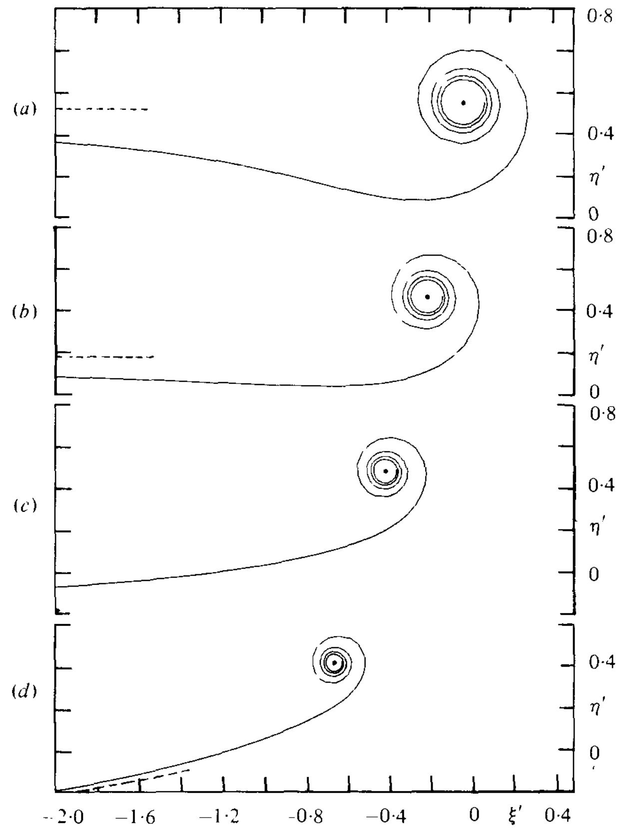

FIGURE 1. ——, self similar vortex sheet shapes in $\rho = \xi' + i\eta'$ plane, single-branch solutions, $\delta = 0$, (a) $p = 0{\cdot}05$, (b) $p = 0{\cdot}333$, (c) $p = 0{\cdot}667$, (d) $p = 1{\cdot}0$. ---, asymptotic shape, $\xi' \to -\infty$ (out of range for (c)).

2.3. *Numerical solution*

As in I, we divide the vortex sheet in the ω plane into three sections and approximate (6) in each, the sections being as follows:

(i) A section described by (8) and defined by $\infty > \lambda \geqslant \lambda_0 > 0$. This section may be regarded as that portion of the sheet which is remote from the spiral.

(ii) An intermediate section defined by $\lambda_0 > \lambda \geqslant \lambda_N > 0$, which includes a substantial portion of the rolled-up sheet. This part of the sheet is further divided into N straight subsections.

(iii) An inner part $\lambda_N > \lambda > 0$ over which we assume that the sheet approximates the tightly wound asymptotic spiral of (12).

Now when $p = \frac{1}{2}$ and $\delta = 0$, we can express the contribution from section (i) of the sheet, to the integral in (6), in closed form (equation (12*b*) in I); this follows from (7). But when $p \neq \frac{1}{2}$ and $\delta = 0$ and for all p at $\delta = 1$, (8) suggests that the sheet may deviate significantly from its original planar position, even for large λ, and it is no longer possible to express this contribution in simple closed form. For the present work then (8) was used directly to determine section (i)'s contribution to the integral which was evaluated numerically, after a suitable transformation, by 32-point Gaussian quadrature. Note that for $\delta = 0, \omega(0)$ is unknown and was fixed in I by satisfying an

p	ξ'_v	η'_v	ϵ	α_p	β_p
0·05	−0·0340	0·3518	−1·8	$9{\cdot}15\times10^{-7}$	1·04
0·1	−0·0667	0·3809	−1·9	$8{\cdot}70\times10^{-4}$	1·12
0·2	−0·1290	0·4274	−2·3	$2{\cdot}39\times10^{-2}$	1·31
0·3	−0·1889	0·4600	−2·6	$6{\cdot}53\times10^{-2}$	1·52
0·333	−0·2084	0·4691	−2·7	$7{\cdot}80\times10^{-2}$	1·60
0·4	−0·2477	0·4792	−3·0	0·101	1·75
0·5	−0·3062	0·4875	2·7	0·124	2·02
0·6	−0·3671	0·4883	2·1	0·135	2·33
0·667	−0·4105	0·4856	1·5	0·138	2·56
0·7	−0·4312	0·4808	1·3	0·139	2·68
0·8	−0·5001	0·4656	0·30	0·137	3·07
0·9	−0·5744	0·4409	−0·87	0·132	3·54
1·0	−0·6564	0·4185	−3·1	0·125	4·02

TABLE 1. Positions of spiral centre and values of constants in small λ asymptotic solution for single-branched solutions, $\delta = 0$.

approximate integrated form of (6) (equation (17) in I) in $(0, \lambda_N)$. This carries over to the present work for $\delta = 0$. For the double-branch case, antisymmetry requires that $\omega(0) = 0$, which is consistent with the vanishing of the present equivalent of (17) in I over section (iii) of the sheet for $\delta = 1$. The solution to (10) for $\delta = 0, p = 1$ was obtained as for (6) but using (11) to evaluate the contribution to the integral on the right side, from section (i) of the sheet.

3. Results and discussion

3.1. *Single branched spirals*

Calculations were performed for values of p in the range $0{\cdot}05 < p \leqslant 1{\cdot}0$. In all cases $N = 97$. The spiralled sheet shapes in the ρ-plane for various values of p are shown in figure 1 while table 1 gives the vortex centre positions $\rho(0) = \xi'_v + i\eta'_v$ in addition to estimates for α_p, ϵ and β_p. The results for β_p are considerably different from those found using the Betz approximation. This is possibly because the effect of the induced velocity from distant parts of the sheet on the rate of change of angular momentum within circles centred on the spiral centre, increases with increasing p (see equation B4 of Moore & Saffman), which is consistent with the present solutions. In the Betz approximation, however, this effect is neglected. The motion of the semi-infinite sheet with $p \leqslant 1$ can be related to the small t behaviour of corresponding finite vortex sheets of the type shed by a general lifting wing. The details of this relationship are given in the appendix.

3.2. *Double branched spirals*

For the infinite plane sheet, detailed calculations were performed for $p = 0{\cdot}05$, $0{\cdot}3$, $0{\cdot}5$, $0{\cdot}7$ and $0{\cdot}95$. In figure 2, which shows selected sheet shapes (both branches) in the $\omega = \xi + i\eta$ plane, it may be seen that for $p = 0{\cdot}05$ and $0{\cdot}5$, the inner rolled-up core closely approximates the axisymmetric spiral given by (12). As $p \to 1$ the solution shows increasing ellipticity. For this reason no ϵ entry is given in table 2 for $p = 0{\cdot}7$ and $0{\cdot}95$. The limit $p \to 1$ corresponds to that discussed by Jiminez (1977, 1980) who reported considerable difficulty in seeking a solution to (6). In fact Jiminez was not

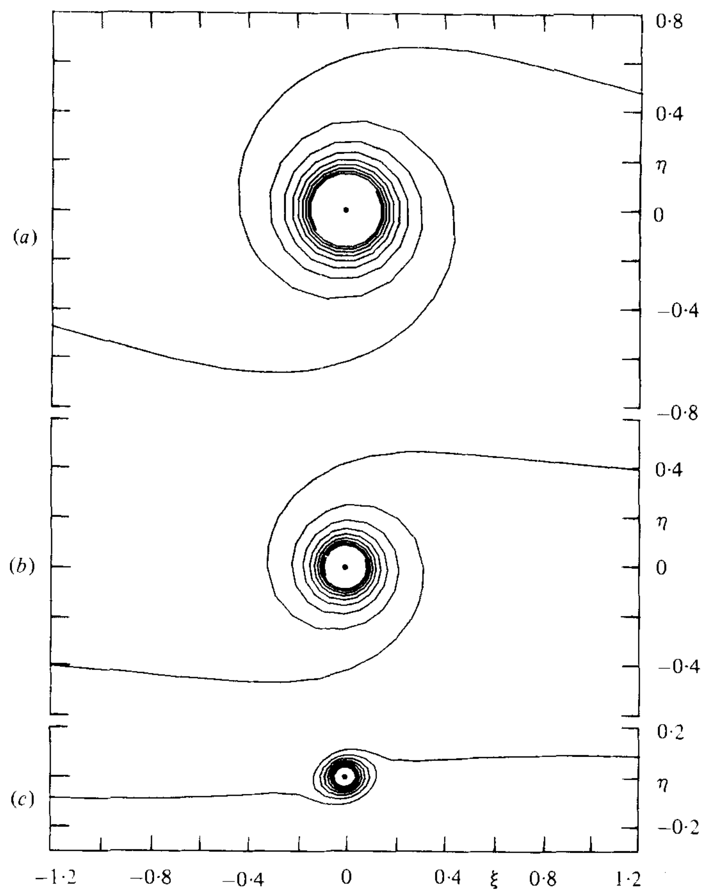

FIGURE 2. ——, self similar vortex sheet shapes in $\omega = \xi + i\eta$ plane, double-branch solutions, $\delta = 1$. (*a*) $p = 0{\cdot}05$, (*b*) $p = 0{\cdot}5$, (*c*) $p = 0{\cdot}95$.

p	ϵ	α_p	β_p
0·05	−1·6	$9{\cdot}78 \times 10^{-7}$	0·975
0·30	−2·2	$8{\cdot}81 \times 10^{-2}$	1·13
0·50	−2·9	0·194	1·30
0·70	—	0·245	1·51
0·95	—	0·22–0·24	2·0–2·1

TABLE 2. Values of constants in small λ asymptotic solution for various p, double-branched solutions, $\delta = 1$.

able to obtain a solution in the sense of achieving monotonic convergence of an algebraic approximation to (6) but rather could only minimize a measure of the error for the approximation. Similarly we found that solutions became increasingly difficult to obtain as $p \to 1$ with none being found for $p > 0{\cdot}95$. In figure 2 it may be seen that the rolled-up double-spiral in the ω plane rapidly reduces in size as $p \to 1$. This may suggest that in the $p \to 1$ limit the spiral reduces to zero size so that the solution degenerates to a trivial but exact and non-singular solution $\omega = -\lambda/2$ representing the initial flat sheet. This would explain difficulties near $p = 1$ since the finite-difference formulation of (6) (see I) implicitly assumes the existence of a rolled-up portion. Thus, our results indicate that the unique spiral-type solution sought by Jiminez as a model for plane shear layer instability, does not exist. Further evidence for this conclusion has been given by Moore (1979) who has shown that the evolution of a plane vortex sheet of constant strength exhibits severe pathological behaviour.

The Betz approximation may be readily applied to the double-branched spiral by assuming that the impulse of vorticity of a section of the infinite sheet initially in $(-x, x)$ about its vortical centre of gravity, remains constant during the roll-up process. Retaining the definition of β_p given in (13), the result is simply $\beta_p = 1$ for all p. This is in surprisingly poor agreement with the results of the present calculations given in table 2 as one might have expected some cancellation between the two branches, of the effects of the unrolled portion of the sheet, on the rate of angular impulse in the core. Note that for the double-branch spiral, the centre of vorticity of the portion of the sheet under consideration ($z = 0$) remains stationary during roll-up in contrast to the behaviour of the single spiral. In the rolled-up core the asymptotic circulation distribution is

$$\Gamma(r) = 4a(\beta_p r)^p,$$

being the sum of one half of this quantity for each spiral arm.

Appendix. Initial tip behaviour of a finite vortex sheet.

Consider a finite, initially flat vortex sheet of length L in $-L \leqslant x \leqslant 0$, which is free to move under its own self induction for $t \geqslant 0$. At $t = 0$, the sheet circulation distribution $\Gamma(x)$ is assumed to be asymptotic to (1) for some p as $x \to 0$, so that a corresponding semi-infinite sheet can be defined as that described in § 2.1 for the same p. Denoting the finite sheet motion by $z(\Gamma, t)$, a relationship between the initial motions of the finite and semi-infinite sheets may be obtained by applying (2) independently to $z(\Gamma, t)$ and $z_0(\Gamma, t)$ at $t = 0$, and subtracting the results to yield,

$$\left[\frac{\partial}{\partial t}(\bar{z} - \bar{z}_0)\right]_{t=0} = \frac{i}{2\pi} \int_0^L \frac{\gamma(u)\, du}{|x| - u} - iap \cot(p\pi)\, |x|^{p-1}, \tag{14}$$

where $\gamma(u) = d\Gamma(u)/du$ with $u = |x'|$. After isolating the singularity in the integral and rearranging the result, we have

$$\left[\frac{\partial}{\partial t}(\bar{z} - \bar{z}_0)\right]_{t=0} = \frac{i}{2\pi} \int_0^L \frac{\gamma - 2apu^{p-1}}{|x| - u}\, du - \frac{iap}{\pi} \int_L^\infty \frac{u^{p-1}}{|x| - u}\, du. \tag{15}$$

Fixing attention on the motion of the finite sheet near the tip $|x| = 0$, it follows from (4) and (15) that for small t

$$\bar{z}(0, t) = \bar{\omega}(0)\, (at)^{1/(2-p)} + \left[\frac{\partial}{\partial t}(\bar{z} - \bar{z}_0)\right]_{\substack{x=0\\t=0}} t + \text{higher order terms.} \tag{16}$$

The first term in (16) is the initial tip motion induced by the γ singularity while the second is that due to the rest of the γ distribution. As $p \to 1$ both terms become singular. The singularities cancel, however, since using (9) and (15) and letting $p \to 1$ gives

$$\bar{z}(0,t) = at\left[\bar{\rho}(0) - \frac{i}{2\pi}\int_0^L \left(\frac{\gamma}{a} - 2\right)\frac{du}{u}\right] + \text{higher order terms.} \tag{17}$$

In (17) the infinite tip velocity due to distant regions of the semi-infinite sheet (see § 2.2 (ii)) has effectively been subtracted and replaced by that due to the finite sheet. The near-tip roll-up is thus in general described initially by the similarity solution, with the appropriate β_p, while the spiral centre moves according to (16) or (17). The $p = 1$ limit is the relevant solution for trailing vortex development from a delta wing at small incidence. It may be shown to be the trailing vortex equivalent to the steady conical flow solution for delta wing leading-edge vortex formation at high angles of incidence (Smith 1966). For a near tip circulation distribution given by (1) with $p > 1$, examination of the behaviour of the first term on the right of (14) as $x \to 0$ shows that the tip velocity singularity vanishes, and there is thus no tendency for tip roll-up.

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